Problem 1. (inspired by TOI 2019) Let $\{x_i\}_{i=1}^n$ be a non-decreasing sequence of real numbers and $\{y_i\}_{i=1}^n$ a non-increasing sequence of real numbers. Let $f : \{x_i\}_{i=1}^n \to \{x_i\}_{i=1}^n$ and $g : \{y_i\}_{i=1}^n \to \{y_i\}_{i=1}^n$. Show that

$$\sum_{i=1}^{n-1} |f(x_i) + g(y_i) - f(x_{i+1}) - g(y_{i+1})| \geq \max_{1\leq j\leq n}\{x_j + y_j\} - \min_{1\leq j\leq n}\{x_j + y_j\}.$$

Solution. Let $S = \sum_{i=1}^{n-1} |f(x_i) + g(y_i) - f(x_{i+1}) - g(y_{i+1})|$ and $a_i = f(x_i)+g(y_i)$ for $i = 1, 2, \ldots, n$. Then,

$$S = \sum_{i=1}^{n-1} |a_i - a_{i+1}|.$$

We will prove that S is minimized if and only if $\{a_i\}_{i=1}^n$ is monotonic. In this case,

$$\min\{S\} = \max_{1\leq i\leq n}\{a_i\} - \min_{1\leq i\leq n}\{a_i\}.$$

We use induction to prove this. The base cases $n = 1$ and $n = 2$ are trivial. For the inductive step, assume the result holds for $n = k$ and consider $n = k + 1$. Let

$$a_r = \max_{1\leq i\leq k+1}\{a_i\}.$$

Since $r - 1$ and $k + 1 - r \leq k$, we can write:

$$S \geq (\max_{1\leq i\leq r-1}\{a_i\} - \min_{1\leq i\leq r-1}\{a_i\}) + |a_{r-1} - a_r| + |a_r - a_{r+1}| + (\max_{r+1\leq j\leq k+1}\{a_j\} - \min_{r+1\leq j\leq k+1}\{a_j\}).$$

Simplify further:

$$S \geq (\max_{1\leq i\leq r-1}\{a_i\} - a_{r-1}) + (\max_{r+1\leq j\leq k+1}\{a_j\} - a_{r+1}) + (a_r - \min\{\min_{1\leq i\leq r-1}\{a_i\}, \min_{r+1\leq j\leq k+1}\{a_j\}\}).$$

Thus,

$$S \geq a_r - \min_{1\leq i\leq k+1}\{a_i\}.$$

Finally,

$$S \geq \max_{1\leq i\leq k+1}\{a_i\} - \min_{1\leq i\leq k+1}\{a_i\}.$$

By induction, the result holds for all n. Hence,

$$S = \sum_{i=1}^{n-1} |f(x_i) + g(y_i) - f(x_{i+1}) - g(y_{i+1})| \geq \max_{1\leq j\leq n}\{x_j + y_j\} - \min_{1\leq j\leq n}\{x_j + y_j\}.$$