(China TST 2016) Let P be a finite set of primes, A an infinite set of positive integers, where every element of A has a prime factor not in P. Prove that there exist an infinite subset B of A, such that the sum of elements in any finite subset of B has a prime factor not in P.

Solution by Wit

Let P' be the set of all positive integers whose all prime divisors are in P.

Claim

For each $K \in \mathbf{Z}$, there is a finite number of (m, m') pairs that satisfy

$$m - m' = K.$$

Proof. Write $m = au^3$ and $m' = bv^3$, where $a, b \in P'$. By Thue's theorem, there are only finitely many integer solutions to $au^3 - bv^3 = K$, and thus we complete our claim.

Remark. In fact, Thue's theorem asserts that for any algebraic number α having degree $d \geq 3$ and for any $\varepsilon > 0$, there exist only finitely many coprime integers p, q with $q > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| < q^{-(d+1+\varepsilon)/2}.$$

We call a subset S of A *spectacle* if and only if the sum of elements in any finite subset of S is not in P'. We need to show that there is an infinite *spectacle* set. We will prove this by induction on the number of $|B|$.

Since the base case $(N = 1)$ seems trivial, we focus on the inductive step and assume that there are $a_1, a_2, \ldots, a_{N-1}, a_N$ contained in B.

To achieve the hypothesis, it comes with the following two cases:

First, there is another element $t \in A$ such that $B \cup t$ is *spectacle*, hence we are done.

Second, there is no such number in $A \setminus B$. In other words, for any $t \in A \setminus B$, there exists $u \subset B$ such that

$$\sum_{i \in u} a_i + t \in P' \iff t = \prod_{j=1}^{m} p_j^{\alpha_i} - \sum_{i \in u} a_i, \quad \alpha_i \in \mathbf{Z}.$$

By PGH, there is a number C such that $x - C \in P'$ for infinitely many $x \in A \setminus B$. Let A' be the set of such x. We now shift A to A'. As a result of our assumption, there is a *spectacle* N-element subset X. Let it be $x_1, x_2, \ldots, x_{N-1}, x_N$. Likewise, there is a number l such that $q - l \in P'$ for infinitely many $q \in A'$. Plugging $q = m + c$, $m \in P'$, we get

$$m - l + C \in P' \iff m - m' = K,$$

for infinitely many $m, m' \in P'$ and K is a constant integer.

However, this contradicts the lemma. Therefore, there is a *spectacle* infinite subset of A, as desired.

Comment. The statement and result are fascinating. I tried to apply such an induction and then found some assumptions by observing something special. However, I was stuck on an unknown theorem (Thue's theorem). The difficulty of this problem is around 30–35 M. Surprisingly, it is China TST P6.