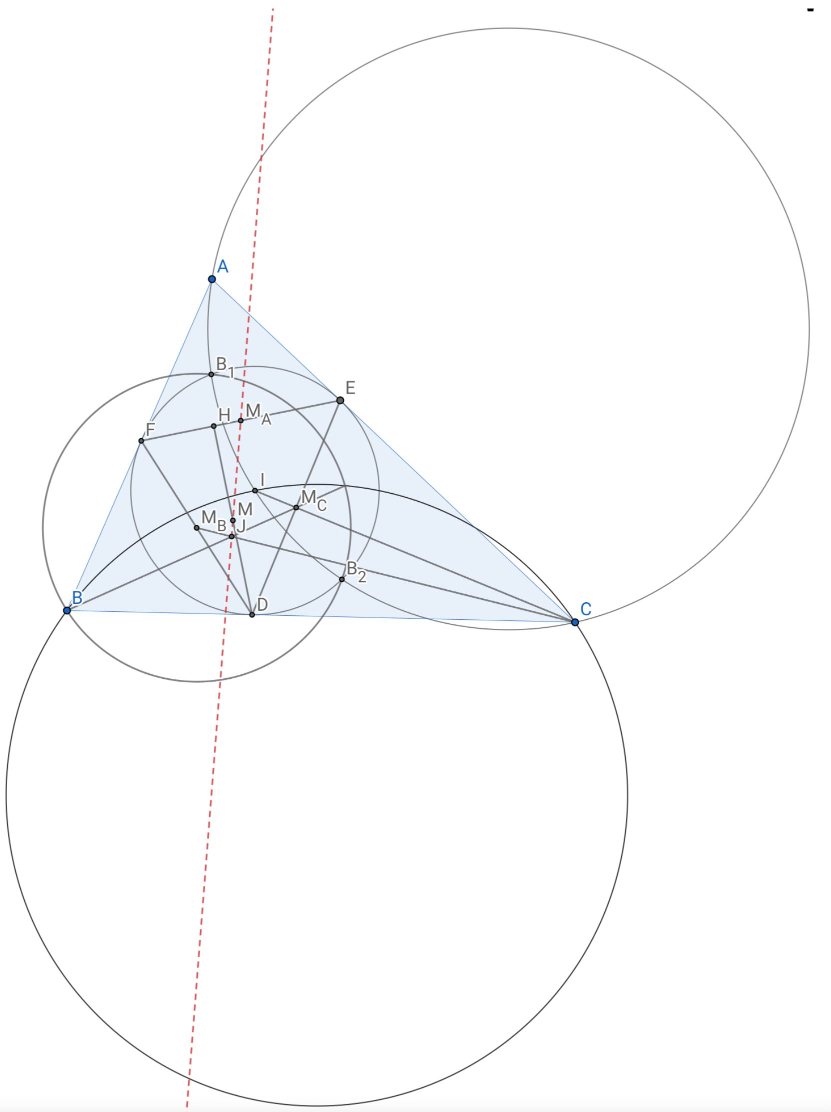
Let ABC be a triangle with incenter I, and whose incircle is tangent to BC, CA, and AB at D, E, and F, respectively. Let K be the foot of the altitude from D to EF. Suppose that the circumcircle of $\triangle AIB$ meets the incircle at two distinct points C_1, C_2, while the circumcircle of $\triangle AIC$ meets the incircle at two distinct points B_1, B_2. Prove that the radical axis of the circumcircle of $\triangle BB_1B_2$ and $\triangle CC_1C_2$ passes through the midpoint M of $\overline{DK}$.

Solution by Nitiwit To solve this problem, we may find other points on the radical axis of $\triangle BB_1B_2$ and $\triangle CC_1C_2$, and then prove that three specific points are collinear instead. Let M_A, M_B, M_C be the midpoints of $\overline{EF}, \overline{DF}, \overline{DE}$, respectively.



Claim 1

M_A lies on the radical axis of (BB_1B_2) and (CC_1C_2).

The radical axis of (AIC), (I), and $(AIEF)$ is the concurrency of AI, EF, and B_1B_2, which is M_A. Therefore, M_A lies on the radical axis of (I) and (BB_1B_2). Similarly, M_A lies on the radical axis of (I) and (CC_1C_2), so M_A lies on the radical axis of (BB_1B_2) and (CC_1C_2).

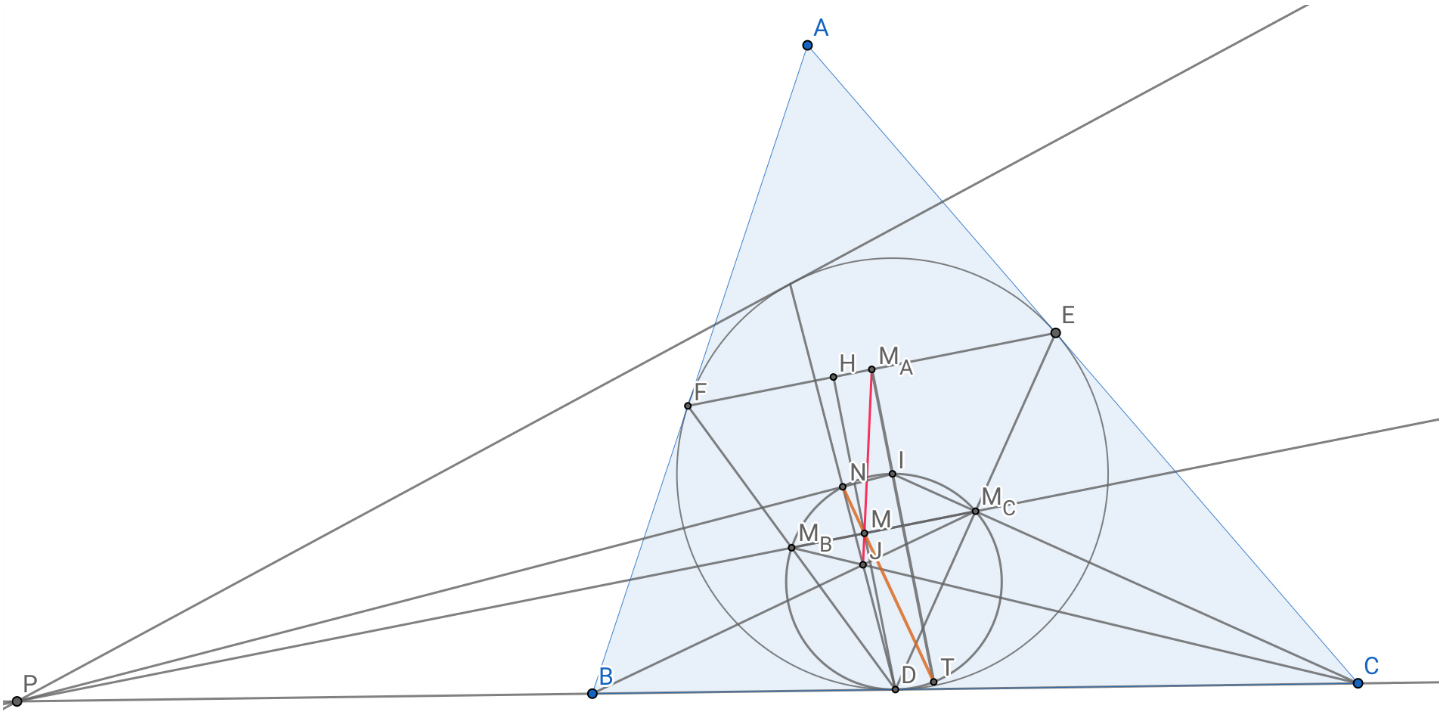
Claim 2

The point of intersection of M_CB and M_BC lies on the radical axis of (BB_1B_2) and (CC_1C_2).

Proof. By the first claim, we also know that M_C lies on the radical axis of (BB_1B_2) and (BIC). Hence, M_CB is the radical axis of (BB_1B_2) and (BIC). By the radical axis theorem applied to (BIC), (BB_1B_2), and (CC_1C_2), we conclude that M_CB intersects M_BC on the radical axis of (BB_1B_2) and (CC_1C_2).

Suppose $M_CB \cap M_BC = J$.

Now, we need to show that M_A, M, and J are collinear.



Since $\frac{M_BM}{M_CM} = \frac{FH}{HE} = \frac{BD}{DC}$ and M_B, M_C, B, C are concyclic, we have $\angle JMM_B = \angle JDB$. We need to prove that $\angle MM_AH = \angle JDB$.

Let DJ intersect (ID) again at N. Let T be the reflection of M_A across M_BM_C. Since D, P, M, N are concyclic, we get $\angle DPM = \angle DNM$. Suppose that $NM \cap (ID) = T'$. We know that $\angle DT'N = \angle PDN$, so $\angle(T'A, AX) = \angle(AX, M_BM_C)$. In other words,

$\overline{DT'} \parallel \overline{M_B M_C} \parallel \overline{DT}$, which means $T = T'$.