# Functional Equation

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January 26, 2025

# 1 Tools

## 1.1 Basic Tools

• Substitutions & Additional Variables → Symmetrization
Sometimes, we have functional equations such that one side of them is symmetric in $x, y$, or we can obtain such a condition by an appropriate substitution. Then, swap $x$ with $y$ to get a new condition, which might prove helpful.
• Iterations and Recurrence Relations

## 1.2 Cauchy + Continuous = Linear

Suppose $f : R \to R$ satisfies $f(x+y) = f(x) + f(y)$. Then $f(qx) = qf(x)$ for any $q \in Q$. Moreover, $f$ is linear if any of the following are true:
• $f$ is continuous on any interval.
• $f$ is bounded (either above or below) in any nontrivial interval.
• There exist $(a, b)$ and $\epsilon > 0$ such that $(x-a)^2 + (f(x)-b)^2 > \epsilon$ for every $x$ (i.e., the graph of $f$ omits some disk, however small).

## 1.3 Visualize the Function

• Drawing the function on a coordinate plane.
• Translate to Directed Graph
e.g., $x \to f^1(x) \to f^2(x) \to f^3(x) \to \ldots \to f^N(x) \to \ldots$ and
$f(1) \to f(2) \ \to \ldots \to f(N) \ \to \ldots$
You may think about:
- Does $G$ contain a cycle?
- Are the number of connected components in the graph finite (or infinite)?
- Is there a finite connected component graph?

## 2 Problems

1.(USA TST 2023) Let $N$ denote the set of positive integers. Fix a function $f : N \to N$ and for any $m, n \in N$ define

$$\Delta(m,n) = \underbrace{f(f(\ldots f}_{f(n) times}(m)\ldots)) - \underbrace{f(f(\ldots f}_{f(m) times}(n)\ldots)).$$

Suppose $\Delta(m,n) \neq 0$ for any distinct $m, n \in N$. Show that $\Delta$ is unbounded, meaning that for any constant $C$ there exists $m, n \in N$ with $|\Delta(m,n)| > C$

2.(USA TST 2019)We say that a function $f : Z_{\geq 0} \times Z_{\geq 0} \to Z$ is great if for any nonnegative integers $m$ and $n$,

$$f(m+1,n+1)f(m,n) - f(m+1,n)f(m,n+1) = 1.$$

If $A = (a_0, a_1, \ldots)$ and $B = (b_0, b_1, \ldots)$ are two sequences of integers, we write $A \sim B$ if there exists a great function $f$ satisfying $f(n,0) = a_n$ and $f(0,n) = b_n$ for every nonnegative integer $n$ (in particular, $a_0 = b_0$). Prove that if $A$, $B$, $C$, and $D$ are four sequences of integers satisfying $A \sim B$, $B \sim C$, and $C \sim D$, then $D \sim A$.

3.(USA TST 2018) Find all functions $f \colon Z^2 \to [0,1]$ such that for any integers $x$ and $y$,

$$f(x,y) = \frac{f(x-1,y) + f(x,y-1)}{2}.$$

4.(IMO Shortlist 2007) Let $c > 2$, and let $a(1), a(2), \ldots$ be a sequence of nonnegative real numbers such that $a(m+n) \leq 2 \cdot a(m) + 2 \cdot a(n)$ for all $m, n \geq 1$, and $a\left(2^k\right) \leq \frac{1}{(k+1)^c}$ for all $k \geq 0$. Prove that the sequence $a(n)$ is bounded.

5.(China National Olympiad 2021) Find $f : Z_+ \to Z_+$, such that for any $x, y \in Z_+$,

$$f(f(x)+y) \mid x + f(y).$$

6.(China TST 2021)Find all functions $f : Z^+ \to Z^+$ such that for all positive integers $m, n$ with $m \geq n$,

$$f(m\varphi(n^3)) = f(m) \cdot \varphi(n^3).$$

Here $\varphi(n)$ denotes the number of positive integers coprime to $n$ and not exceeding $n$.

7.(China TST 2019)Find all functions $f : R^2 \to R$, such that
1) $f(0,x)$ is non-decreasing ;
2) for any $x, y \in R$, $f(x,y) = f(y,x)$ ;
3) for any $x, y, z \in R$, $(f(x,y) - f(y,z))(f(y,z) - f(z,x))(f(z,x) - f(x,y)) = 0$;
4) for any $x, y, a \in R$, $f(x+a, y+a) = f(x,y) + a$ .

8.(China TST 2012) $n$ being a given integer, find all functions $f\colon Z \to Z$, such that for all integers $x, y$ we have $f\left(x + y + f(y)\right) = f(x) + ny$.

9.(China TST 2009) Consider function $f : R \to R$ which satisfies the conditions for any mutually distinct real numbers $a, b, c, d$ satisfying $\frac{a-b}{b-c} + \frac{a-d}{d-c} = 0$, $f(a), f(b), f(c), f(d)$ are mutully different and $\frac{f(a)-f(b)}{f(b)-f(c)} + \frac{f(a)-f(d)}{f(d)-f(c)} = 0$. Prove that function $f$ is linear

10.(Putnam 2022)Find all continuous functions $f : R^+ \to R^+$ such that

$$f(xf(y)) + f(yf(x)) = 1 + f(x + y)$$

for all $x, y > 0$.